The Quantum Hall Effect

글_Hantao Lu · 전 고등과학원 물리학부 연구원

Three decades have passed since the discovery of quantum Hall effect (QHE) by von Klitzing in 1980[1]. But many subjects in this field are still on the front of condensed matter physics. In this small article, we will give a brief description of some features of QHE, hoping to bring some light to its unfathomable beauty and richness.

QHE happens when we put a two-dimensional (2D) electron gas under a strong magnetic field at low temperature (There are various ways to prepare a 2D electron gas, from traditional semiconductor heterostructures to today's graphene. Here we will focus on the former, i.e. the QHE in non-relativistic 2D gases). A series of plateaus of Hall resistance, defined as the ratio of the transverse voltage drop and longitudinal current $R_H=V_H/I$ (see also Fig. 1), can be observed with the increase of magnetic field. Associated with the appearance of plateaus, the longitudinal resistance always drops towards zero (Fig. 2).

The quantizied Hall resistance can be formulated as

$$R_H = \frac{1}{\nu}\frac{h}{e^2}.$$

Here, we encounter two important quantities in QHE: One is the von Klitzing constant $R_K=h/e^2$, which has the unit of resistance, a fact that was little appreciated prior to the discovery of QHE. The other is the filling factor ν, the density ratio of electrons and flux quanta of the applied field. As we know now, the filling factors of all discovered quantum Hall states are exactly given by some rational numbers, such as ν=1, 2, 1/3, 2/5,..., as shown in Fig. 2. A Hall state is fully characterized by its filling factor.

The case for integral fillings, i.e. ν=1, 2,... (dubbed integer QHE, IQHE), as we well know, can be easily understood from the occurrence of Landau levels. As shown in Fig. 3, due to the unavoidable presence of impurities in samples, the Landau levels are usually broadened into extended and

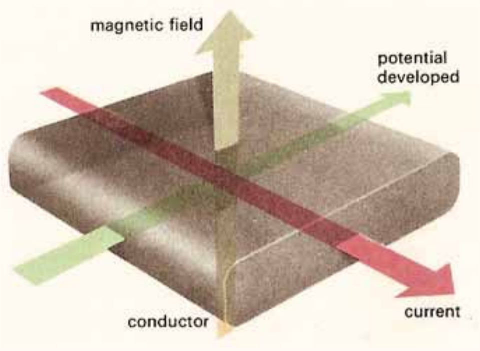

Figure 1. Schematics of magnetotransport measurement in quantum Hall effect.

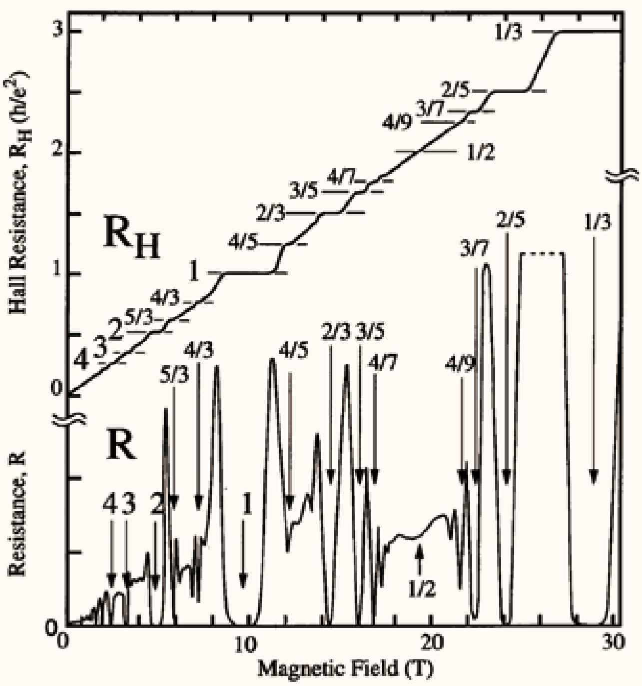

Figure 2. Overview of Hall and longitudinal resistances. Source: H. L. Stormer, *Rev. Mod. Phys*. **71**, 875-889 (1999).

localized states. A plateau will appear when the Fermi level happens lying in localized regions. Prominently, the magnitude of Hall conductance (reciprocal of Hall resistance), in the unit of $R_K^{-1}=e^2/h$, turns out exactly to be the number of occupied bands.

This is not trivial. The quantization of Hall resistance, the value of which only depends on the fundamental constants of physics, is highly accurate (up to 10^{-9}), universal and robust: It is independent of the sample type, geometry, materials parameters, also immune to (weak) disorders pervaded in condensed matter systems. There are reasons for it. The quantized resistance of IQHE is closely related to the gauge invariance of the system, and can be expressed as the sum of the first Chern numbers of the occupied bands[2]. The robustness of QHE can be attributed to this topological stability. In the classification theory of topological insulators and superconductors[3], IQHE belongs to class A (unitary class), whose space of quantum ground states in 2D is partitioned into topological sectors labeled by an integer (here the filling factor).

One feature of quantum Hall states is their featureless: different quantum Hall liquids have the same symmetries. Thus we cannot use symmetries

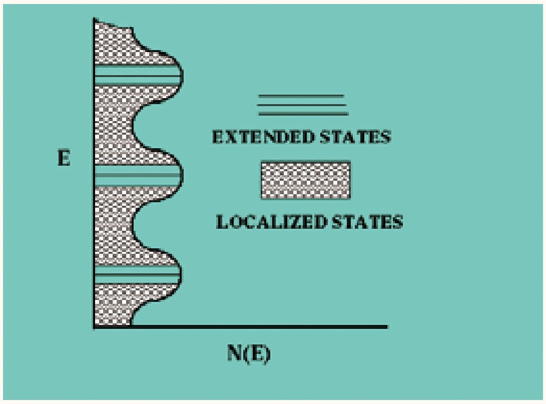

Figure 3. Diagram of Landau levels, stolen from Prof. Yiming Qiu's homepage.

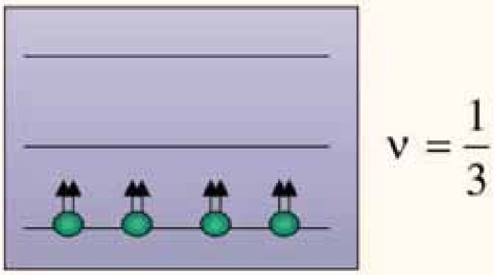

Figure 4. The composite fermion picture for $\nu=\frac{1}{3}$ FQHE. After every electron captures two flux (vortices) to form a composite fermion, the effective filling factor turns to be 1.

and local order parameters to distinguish different Hall states liquids. We have already mentioned in IQHE, Chern numbers can be used for characterization. Recognizing hidden topological order becomes more salient and indispensable when we encounter the fractional quantum Hall effect (FQHE).

The first FQHE was discovered in 1982, not long after the finding of IQHE[4]. The observation of quantizied plateaus of Hall resistance when a Landau level is not fully occupied was truly a surprise. Considerable efforts have been made and the mist over it began to gradually retreat after Laughlin's seminar work[5].

The physical picture is not hard to grasp. Since the kinetic energy is totally suppressed due to the formation of Landau levels (Here we solely focus on the single top no-empty level and ignore the level mixing), Coulomb interaction in turn plays a crucial role. Electrons try to find some way to settle down as the partly-filled level permits them to play tricks. In FQHE, the electrons take some actions to organize themselves by reconciling their desire to order (order in real space, to minimize Coulomb repulsion) with the difficulties introduced by frustration (aroused by the magnetic field). The outcome is a quantum liquid. Such steps can be visualized as the formation of composite fermions: a novel bound state of one electron with even number (say, $2p$) vortices[6]. So, as the fermions move around, the attached vortices will produce Berry phases, canceling part of the Aharonov-Bohm phases originating from the field. Consequently, they effectively experience much weaker field and less frustration. Further, importantly, because considerable part of the Coulomb interaction has been used in making them, the residual interaction between these new-type particles is weak now. At last, it turns out that nearly all FQHE can be categorized into IQHE of composite fermions (except one or two instances with peculiar even-denominator filling factors, as 5/2 or 7/2, which can be understood as pairing states of composite fermions). This can be summarized as:

$$v = \frac{v^*}{2pv^* \pm 1},$$

where v is the filling factor of electrons, and v^* is the corresponding one referring to composite fermions, usually an integer.

When the theory of composite fermions was proposed, nearly at the same time, the idea of vortices attachment also leaded to another fruit: An effective-field-theory description of FQHE, a theory with Chern-Simons terms $\sim\epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$[7]. The fact that the term is called topological term means it does not know much about clocks and rulers − the fine structure of the manifold where it lives, but sensitive to its global properties (topology of the manifold). Such quality makes the theory especially suitable for studying the topological orders in FQHE.

Different from the traditional definition of order, topological order can not be labeled by local order parameters. But it can manifested itself in miscellaneous ways. In FQHE, one way is the degeneracy of the ground states: It depends on the topology of the spacetime manifold. For example, the $v=1/q$ FQHE has q^g degenerate ground states on a Riemann surface of genus g. The others, closely related, are fractional charges and statistics of quasiparticles. Such properties can be explicitly demonstrated in the effective theory. Furthermore, the bulk properties of FQHE, if described by the Chern-Simons theory, have close relation to the edge, which is characterized by a conformal field theory (CFT)[8]. Specifically, the bulk wavefunction can be reproduced by the conformal blocks of the CFT. At the same time, the boundary theory which is concomitant with the Chern-Simons theory is precisely the CFT of the edge. Chiral gapless modes, known as edge states, appear at the boundary between a quantum Hall droplet and the vacuum, or between different quantum Hall states with distinct filling factors, as a consequence of the change of the topological order. The correspondence between the topological properties of the gapped bulk and the gapless surface degrees of freedom is one of the examples of so-called "holographic principle", which claims, roughly speaking, that we can read out the information contained in a volume by inspecting the information that resides on the boundary of that region.

As the last part of the article, we would like to mention a bit about $\nu=5/2$ state: the mysterious FQHE with an even denominator. Although over 20 years have passed by since its discovery in 1987[9], the studies on it have never stopped and even intensified in recent years because of its possible application to fault-tolerant quantum computation[10]. It has been widely accepted now that the ground state of the 5/2 is a kind of p-wave (more precisely, $p\pm ip$) pairing state with polarized electrons. As a consequence, the quasiparticles support non-Abelian braiding statistics. A corresponding non-Abelian Chern-Simons theory has also been proposed. But, there are still many question marks remained: Is the 5/2 state a fully polarized state, or partially polarized (There are conflict experiments on it till now)? If so, does it belong to the Pfaffian or anti-Pfaffian class? What's the mechanism of the pairing? Can we bridge the gap between the effective field theory and the original microscopic Hamiltonian?

In conclusion: "There is no end to the richness of the nature." — Xiao-Gang Wen

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