Hamiltonian Fixed Points, Lagrangian Intersections, and Earthquake Miracles

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The goal of this article is to describe the three seemingly unrelated topics of the title and briefly explain how they are faces of the same coin, a so called *leafwise fixed point.*

Hamiltonian Fixed Points

The time evolution of a classical mechanical system is governed by Hamilton's equation derived from a (possibly time-dependent) Hamiltonian (energy) function. This is a first order ordinary differential equation for a point in phase space as a function of time. Such a point corresponds to the (generalized) position and momentum of a given object (e.g. a stone).

Phase space naturally generalizes to a symplectic manifold, i.e., a manifold together with a closed and non-degenerate two-form. An example of a symplectic manifold is a real surface equipped with an area form. The time-one flow of Hamilton's equation is called a Hamiltonian diffeomorphism. In the 1960s V. I. Arnold formulated the following famous conjecture for a closed symplectic manifold:

Every Hamiltonian diffeomorphism has at least as many fixed points as a smooth real-valued function on the manifold must have critical points.

If the Hamiltonian function is periodic in time then the fixed points correspond to periodic trajectories of the system. From a dynamical viewpoint, such trajectories are the nicest ones. If the Hamiltonian diffeomorphism is generated by some time-*independent* function then every critical point of the function is a fixed point of the diffeomorphism. Hence the assertion of the conjecture is true for such a diffeomorphism.

The conjecture has been a major driving force in symplectic geometry. It has lead to the important technique of Floer homology. Variants of it have been

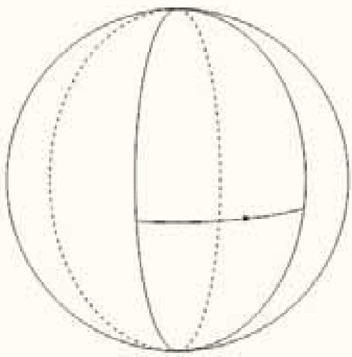

Figure 1. The arrow indicates the Hamiltonian diffeomorphism on the two-sphere given by rotation about the vertical coordinate axis by some angle. Its fixed points are the north and south poles. It is generated by a multiple of the vertical coordinate function.

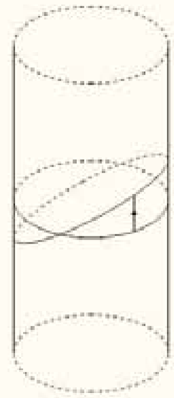

Figure 2. The horizontal central circle is the Lagrangian submanifold of the cylinder corresponding to zero momentum. The arrow indicates a Hamiltonian diffeomorphism, mapping the circle to the tilted ellipse.

proven in many cases. However the original version is still open. (For a brief overview see e.g. Chapter 9.1 [MS].)

As an example consider the symplectic manifold given by the two-sphere together with any area form. The minimal number of critical points of a function on this manifold is two. Hence in this example the conjecture states that every Hamiltonian diffeomorphism on the two-sphere has at least two fixed points. This is known to be true. (See figure 1 for an example.)

Lagrangian submanifolds

A submanifold of a symplectic manifold is called *Lagrangian* if it is half-dimensional and the symplectic form vanishes on vectors tangent to it. An example is the set of all points in phase space with vanishing momentum component. (This set can be identified with configuration space.) Lagrangian submanifolds are interesting, since they are in some sense minimal geometrical objects. The following question has been another main source of developments in symplectic geometry over more than 20 years: *Given a Lagrangian submanifold, what is the minimal number of its intersection points with its image under a Hamiltonian diffeomorphism?*

Non-trivial lower bounds on this number have been found in many examples. The question gave rise to a version of Floer homology for Lagrangian submanifolds, which in turn led to the notion of a Fukaya category. (See [FOOO].) This is a crucial ingredient in homological mirror symmetry.

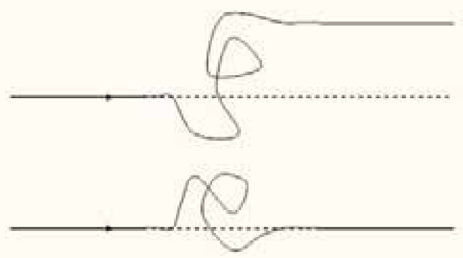

Figure 3. A generic trajectory and a "miracle" for the perturbed system. The straight (solid and dashed) lines refer to unperturbed trajectories.

As a simple example, consider the (infinite) cylinder, i.e., the Cartesian product of the circle and the real line. Equipping it with the standard area form, we obtain a symplectic manifold. (This is the phase space for a particle on the circle.) As Lagrangian submanifold we take set of all points in the cylinder with vanishing momentum (i.e., second component). The intersection of this submanifold with its image under a Hamiltonian diffeomorphism contains at least two points. This follows from an elementary argument. (See figure 2 for an example.)

Earth quake miracles

Consider an autonomous (i.e., time-independent) Hamiltonian system and a time-dependent perturbation, which starts at time zero and ends at time one. We may think of the perturbation as an earth quake. Consider a point in phase space whose perturbed trajectory coincides with its unperturbed trajectory after time one, except for a time shift. This means that, up to a time shift, the system returns to its initial condition after the earth quake. We may call this a "miracle". (See figure 3.) Note that in case of a miracle the total energy of the system is the same for negative times and times bigger than one.

As an example, consider a particle in the plane, which is attached to an anchor via a perfect spring. For negative times and times bigger than one the anchor is fixed at the origin of the plane, while from time zero to one it moves in an arbitrary way. (Note that the unperturbed system is the harmonic oscillator.) Assume that the particle's initial energy is positive. Then for negative times, the particle moves on an ellipse in the plane, with center the anchor. From time zero to one, it is "shaken around" due to the "earth quake", while after time one, it moves again on some ellipse (or is at rest at the origin). In general, the initial and final ellipses may be very different. They may for example correspond to different energy levels.

A "miracle" can be interpreted as a choice of an initial ellipse (and direction and initial point on it), which agrees with the resulting final ellipse (and direction). It follows from the main result in my article [Z1] that generically at least two "miracles" occur on every positive energy level set, provided that the "earth quake" is not too strong.

Common description as leafwise fixed points

Fixed points, Lagrangian intersection points, and "earth quake miracles", as described above, are special cases of leafwise fixed points of a map from a symplectic manifold to itself. Such a point is defined to be a point on a given coisotropic submanifold, whose image under the map lies in its isotropic leaf. (For explanations see [Z1].) (Coisotropic submanifolds arise in mechanics as constraint systems of first class.)

If the coisotropic submanifold is the whole manifold, then leafwise fixed points are true fixed points. Taking it to be Lagrangian (and connected), we obtain Lagrangian intersection points. If it is a level set of an autonomous Hamiltonian function, then leafwise fixed points of a perturbation correspond to "miracles".

Generalizing the previous discussion, we may ask the following question: *Given a coisotropic submanifold, how many leafwise fixed points does a Hamiltonian diffeomorphism at least have?*

Recently, this question has drawn the attention of symplectic geometers. If the coisotropic submanifold is a hypersurface then interesting lower bounds on the number of leafwise fixed points have been found in many situations. (For an overview and references see [Z1].) For a coisotropic submanifold of general dimension the question is largely unexplored. My articles [Z1, Z2] are concerned with this general situation.

In [Z1] I provided sufficient criteria under which the number of leafwise fixed points is bounded below by the sum of the Betti numbers of the coisotropic submanifold. The bound is sharp if the coisotropic submanifold carries a perfect Morse function. In [Z2] I defined the notion of a Maslov map for a coisotropic submanifold and proved a related lower bound. Furthermore, in joint work with Jan Swoboda [SZ1, SZ2] we provide applications of the existence of a leafwise fixed point.

The above question naturally combines a geometric object (the coisotropic submanifold) with a dynamical object (a map from the manifold to itself). Although the objects involved are elementary to define, sophisticated concepts and techniques are needed to answer the above question in interesting examples.

In my opinion this is a beautiful example of how abstraction and generalization can lead to a unification of apparently unconnected matters.

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