Majorana zero modes of two-dimensional chiral *p*-wave superconductors

글 _ Hantao Lu · 고등과학원 물리학부 연구원

As well known, in a superconductor, because of the condensation of Cooper pairs, the magnetic field, if any, will be expelled from it. The phenomenon is called Meissner effect. But in some situations, such as the increase of temperature or magnetic field or the presence of defects or impurities, magnetic field can be trapped inside the superconductor, forming a vortex structure, as illustrated in Fig. 1: within the narrow vortex core region, superconductivity is suppressed or destroyed. The circulating supercurrents outside these cores can induce magnetic field with a quantizied flux. Especially, in two dimensional (2D) space, a vortex can be regarded as a point-like particle. In this article, we will talk about zero modes, a kind of peculiar low-lying excitations localized on vortices of a 2D superconductor.

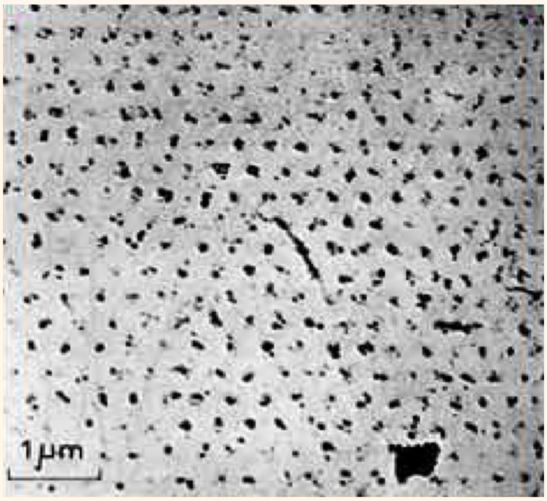

Figure 1: First image of vortex lattices on Pb-4at%In specimen, by U. Essmann and H. Trauble in 1967 (Phys. Lett. 24A, 526 (1967))

Before our showing, I think we need a little preliminary warm-up. Consider a simple Hamiltonian as

$$H=\sum_{ij}c_i^\dagger H_{ij}c_j, \qquad (1)$$

where $c_i^\dagger$ and c_i are fermion creation and annihilation operators, and the index "i" is a label for single-particle states, which may incorporate both the space coordinate (or momentum), and the spin index. We want to diagonalize this Hamiltonian, what should we do? What's the meaning of diagonalizing a Hamiltonian?

Since the Hamiltonian is quadratic, there are many ways to handle it. Here I just show one way, which will be useful for the further discussion. Introduce a new quasi-particle operator $\gamma_\alpha^\dagger=\sum_i u_{i\,\alpha}c_i^\dagger$. By insisting that $[H, \gamma^\dagger_\alpha] = E_\alpha \gamma^\dagger_\alpha$, which implies that

$$H = \sum_\alpha E_\alpha \gamma^\dagger_\alpha \gamma_\alpha + \text{const.} \qquad (2)$$

One obtains the linear equations

$$\sum_j H_{ij} u_{j\alpha} = E_\alpha u_{j\alpha}$$

We can see that by u_α (column index i has been suppressed), which are just the eigenvectors of H_{ij}, the new operators γ_α actually diagonalize the Hamiltonian. Eq.(2) is obtained. The spectra of H can be read directly from the eigenvalues E_α. The quasiparticles γ_α have the same anticommutation relations as the c_i, which is guaranteed by the mutual orthonormality and completeness of the eigenvectors u_α, since H_{ij} is a Hermitian matrix.

After this, let's go to our topic. Consider a Hamiltonian

$$H_p = \int d^2r \left[\psi^\dagger \left(-\frac{\nabla^2}{2m} - \mu \right) \psi + \frac{1}{2} \psi^\dagger [\Delta(r) * (\partial x + i\partial y)] \psi^\dagger + H.c. \right], \qquad (3)$$

where $*$ is the symmetrized product $[A*B = (AB+BA)/2]$, μ is the chemical potential. Eq.(3) is a mean-field BCS-type Hamiltonian, widely used to describe the $p_x + ip_y$ paring between spinless fermions in 2D space (notice $\partial x + i\partial y$ in Eq.(3), that is the reason why we call it $p_x + ip_y$). Here, I try to keep the Hamiltonian form as general as possible. For convenience, I would like to rewrite the Hamiltonian into a matrix form

$$H_p = \int d^2r \Psi^\dagger \mathrm{H} \Psi$$

where Ψ is a two-component spinor: $\Psi^T = (\psi, \psi^\dagger)^T$, and the matrix H

$$\mathrm{H} = \begin{pmatrix} -\frac{\nabla^2}{2m} - \mu & i[\Delta(r) * (\partial x + i\partial y)] \\ i[\Delta^*(r) * (\partial x - i\partial y)] & \frac{\nabla^2}{2m} + \mu \end{pmatrix}. \qquad (4)$$

How to solve it? OK, encouraged by the success of our trick on Eq.(1), we would like to follow the same way. But this time, it is found we have to combine creation and annihilation operators together to make things work. The necessary of doing this can be easily understood from the above matrix representation. So set

$$\gamma^\dagger = \int d^2r [u(r)\psi^\dagger(r) + v(r)\psi(r)]$$

Undoubtedly, when $(u,v)^T$ is a solution of the matrix H with eigenvalue E, we have $[\mathrm{H}, \gamma^\dagger] = \mathrm{E}\gamma^\dagger$. γ is known as Bogoliubov quasiparticle, and the corresponding eigenvalue problem is called Bogoliubov-deGennes (BdG) equation, which is ubiquitous in the literature of superfluids and superconductors.

If we stare Eq.(4) for enough long time, probably we can find that H has some special symmetry:

$$\sigma_1 \mathrm{H} \sigma_1 = -\mathrm{H}^*, \qquad (5)$$

where σ_1 is the first Pauli matrix acting in the 2×2 space of the matrix H. What does it mean? It means that if $\mathbf{v} := (u,v)^T$ is an eigenvector with eigenvalue E, then $\sigma_1 \mathbf{v}^* = (v^*, u^*)^T$ is guaranteed to be an eigenvector of H comes with eigenvalue $-E$. The BdG eigenvalues come in $\pm$ pairs! But in the situation we concern, this is some kind of trivial thing. Noticing that $\gamma^\dagger = u\psi^\dagger + v\psi$ (integral symbol suppressed), it is easy to see $\gamma^\dagger_E = \gamma_{-E}$, we seem to talk about the same thing and no special happens.

But, hold on. What will happen when $E=0$? If $\mathbf{v}$ is a zero mode solution, so does $\sigma_1\mathbf{v}^*$. Taking linear combinations $\mathbf{v}+\sigma_1\mathbf{v}^*$ and $i(\mathbf{v}-\sigma_1\mathbf{v}^*)$ of these degenerate modes, we can always ensure the relation

$$\sigma_1\mathbf{v}^*=\mathbf{v}$$

for every zero mode. That means $u^*=v$, and $\gamma^\dagger=\gamma$. The zero-energy level becomes a self-conjugate (Majorana) fermion. This is the Holy Grail in our small article, mentioned in the title.

But the question is: such zero mode solution exist or not? Symmetry analysis does not give us definite answer about this. Unfortunately, it was found for the vortex-free case, such solution usually does not exist. To make things clear, suppose a uniform order parameter as $\Delta(\mathbf{r})=\Delta_0$, it is easy to find the spectra in momentum space

$$E_k=\sqrt{\left(\frac{k^2}{2m}-\mu\right)^2+|\Delta_0|^2k^2}$$

Since E_k has a gap for all $\mathbf{k}$, with the exception of the critical point at $\mu=0$, there are no zero modes in the absence of vortices. Actually, $\mu=0$ is a critical point separating two phases: weak- and strong-pairing phases with different topology [1]. At present, we just want to stay safely in the weak-pairing regime ($\mu=0$) and have no wish to cross the line.

A kind of disappointment! But patient. As always happens, the Nature, she (yes, we always call "she", not wishing to regard the Nature as some kind of ruthless, unemotional robot), would not likely miss a chance to show her deep beauty and charms. Even sometimes through a subtle way. Things are changed when vortices join in.

Suppose we introduce a single winding number (vorticity) vortex located at the origin. That means, in Eq.(3), the gap function $\Delta(\mathbf{r})$ takes the form as $\exp(i\theta)\Delta(r)$ (azimuthal symmetry assumed), where θ are polar coordinates. There is a dramatic change in the behavior of the system. Something essential: the phase winding number n of a vortex in a superconductor relates to the number of branches of low energy excitations of bound states localized in the vortex core. The relation is guaranteed by an index theorem [2]. The index theorem also works in the model considered here, which describes a spinless (p_x+ip_y) -paired superconductor. Further, it was found odd winding number vortices trap zero-energy bound states in the cores, while for even winding number vortices, quite strangely, there is no such zero-energy mode. I' ve tried to find some simple way to explain the fact. But unfortunately I could not succeed now. One way to think about this is by mapping the 2D vortex problem on an effective 1D problem by performing angular momentum decomposition with respect to the center of the vortex, one can show that for odd winding number vortices there is a unique angular momentum channel supporting such kind of zero modes, analogous to the Jackiw-Rebbi zero modes in 1D Dirac theory, while the fortune does not happen in the even-vorticity case. For the interested readers, I strongly recommend the paper in Refs. [3, 4].

One thing worth mention: Why we have to search Majorana zero modes in *p*-wave superconductors, not simpler *s*-wave? The argument is that: it is well known for *s*-wave pairing, the Bogoliubov quasiparticles $\gamma^{\dagger}=u\psi_{\uparrow}^{\dagger}+v\psi_{\downarrow}$, the spin-up and spin-down components are mixed up, making the self-conjugacy relation impossible.

Why such zero modes, confined in the cores of vortices, attract us so much? Besides naturally scientific curiosity, there is another extra incentive. They may be useful in the development of topological quantum computer, a highly expected candidate for the future revolutionary generations of computer. In order to be useful, usually at least two requirements must be met. First is the system should be immune to errors cause by local perturbations (fault tolerance). Secondly, non-Abelian statistics of quasiparticles should be realized in order to construct logic unit. We will try to justify that the p_x+ip_y model does satisfy the preconditions.

It was shown that the Majorana zero mode enjoys topological protections (in an extended meaning). Not only its existence is guaranteed by an index theorem in some situations, but also as long as the structure and symmetry of the Hamiltonian (Eq.(5)) are preserved, and the perturbation is not strong enough to destroy the vortex (overwhelming the superconducting gap' s protection), this exact fermion zero mode is robust. Of course, vortices should be separated apart quite well to prevent the tunneling between them from lifting the degeneracy.

Imagine a system of isolated vortices with one Majorana fermion for each core. Majorana fermions can be combined into complex fermionic operators. The rule like this:

$$\Psi=\frac{\gamma_1+i\gamma_2}{2},\qquad \Psi^{\dagger}=\frac{\gamma_1-i\gamma_2}{2}$$

Each fermionic level may be either filled or empty, giving rise to the degeneracy of the ground state equal to 2^n (remember the Majorana fermions have zero energy). Consider now a braiding operation T_i, which interchanges the neighboring vortices γ_i, γ_{i+1}. This results a *nontrivial* transformation rule as:

$$T_i:\begin{cases}\gamma_i \mapsto \gamma_{i+1}, \\ \gamma_{i+1} \mapsto -\gamma_i \\ \gamma_j \mapsto \gamma_j \qquad \textit{for } j\neq i \ \textit{ and } \ j\neq i+1.\end{cases}$$

The essential ingredient for what happen here is that the Majorana fermions γ_j *change sign* under a shift of the superconducting phase by 2π. By extending the action of T_i from *operators* to the 2^n-dimensional Hilbert space, one can decide the representation of braid group B_{2n} in this space, which turns out to be unitary and non-Abelian. A comprehensible exposition about this statement can be found in Ref. [5].

You would say, OK, that's funny. But

does such a toy model we're talking about, has any incarnations in the real world, not just a mirage or mind creation? The answer is positive. Hopeful candidates include the layered Sr_2RuO_4 superconductor, spin-polarized cold atoms in optical traps, the *A* phase of ^{3}He films, and the most concrete one till now, the quantum Hall system at filling factor v=5/2. The ground state of 5/2 -filling is widely believed to be descried by the putative Moore-Read Pfaffian wave function. Following the time arrow, the history might be this. Moore and Read in 1991 [6] found that according to their wave function, the quasiparticles of 5/2-state exhibit non-Abelian statistics. Since in physics, the 5/2-state can be regarded as a special spinless p_x+ip_y superconductor of the "composite fermions" (a bound state of one electron and even number of magnetic flux), people may wonder whether this mysterious trace of non-Abelian can also be track down in a *normal* superconductor [1]. As we have shown that, it turned out to be true finally. The carriers of the non-Abelian statistics are *vortices* with odd-vorticity. Equally and happily, a group of superconductors or superfluids (chiral *p*-wave pairing) join the growing club of non-Abelian anyon and share the same glory.

Last little comment: Step by step, the face, once looked so distant and unpredictable, becomes more and more familiar and friendly. Such things not only happen in daily lives, but also in science.

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