Frobenius Manifolds and Orbit Spaces

글_ Dafeng Zuo · 고등과학원 수학부 조교수

In the beginning of 1990's, E.Witten, R.Dijkgraaf, E.Verlinde and H.Verlinde introduced one remarkable system of partial differential equations, that is WDVV equations of associativity, on two dimensional topological field theory (2D TFT). Roughly speaking, the problem is to find a quasihomogeneous function $F = F(t)$ of the variables $t = (t^1, \cdots, t^n)$ such that the third derivatives $c_{\alpha\beta\gamma} := \partial_\alpha \partial_\beta \partial_\gamma F(t)$ for any t are structure constants of an associative algebra $\mathcal{A}_t$ with a t-independent unity.

In the physical setting, the solutions of WDVV equations describe the moduli space of topological conformal field theory. From the point of mathematical view, some particular solutions of WDVV equations with certain good analytic properties are generating functions for the genus zero Gromov-Witten invariants of Käler, and more generally, of symplectic manifolds.

In order to understand a geometrical foundation of 2D TFT on the bases of WDVV equations, in 1993 B.Dubrovin extended the Atiyah' s axioms of TFT for the two-dimensional case by the properties of the canonical modulo space of a TFT model. On this way he invented a nice geometrical object, that is, Frobenius manifold designed as a coordinate-free formulation of WDVV equations. It was shown that any model of 2D TFT was encoded by a Frobenius manifold and many constructions of TFT (integrable hierarchies for the partition function, their bi-hamiltonian formulae and τ-functions, string equations, genus zero recursion relations for correlators) can be deduced from geometry of Frobenius manifolds.

Nowadays the theory of Frobenius manifolds has established remarkable relationships between some rather distant mathematical theories, including the theory of Gromov-Witten invariants, singularity theory, differential geometry of the orbit spaces of reflection groups and of their extensions, the hamiltonian theory of integrable hierarchies, and so on.

In this article we want to review some recent developments about the Frobenius structures on the orbit spaces of reflection groups and of their extensions.

A clue to understanding of a rich differential-geometric structure of the orbit spaces can be found in the singularity theory due to V.I.Anorld and K.Saito etc. According to this the complexified orbit space of an irreducible Coxeter group is bi-holomorphic equivalent to the universal unfolding of a simple singularity. Under this identification the Coxeter group coincides with the monodromy group of vanishing cycles of the singularity. The discriminant of the Coxeter group (the set of irregular orbits) is identified with the bifurcation diagram of the singularity. The invariant Euclidean inner product coincides with the pairing on the cotangent bundle defined by the intersection form of vanishing cycles. The bi-holomorphic equivalence is given by the period mapping.

Additional differential-geometric structures on a universal unfolding of an isolated hypersurface singularity are determined by the Grothéndieck residues. For *ADE* cases the formulae for the residues were rediscovered by E.Witten R.Dijkgraaf, E.Verlinde and H.Verlinde. Moreover they discussed 2D TFT from the point of view of the theory of singularities.

Motivated by these and Anorld's observations, B.Dubrovin defined a monodromy group $W(M)$ for an arbitrary l-dimensional Frobenius manifold M which acts on an l-dimensional linear space. Let us look at two simple examples which are related to topological Laudau-Ginsburg models and topological sigma models respectively.

Example 1. [*W*(*M*)=*Coxeter group* A_1] $l = 1$,

$M = \mathbb{C}$, $t = t^1$,

$$F(t) = \frac{1}{6}t^3, \quad E = t\partial_t, \quad e = \partial_t, \quad \eta^{11} = <\partial_t, \partial_t> = 1.$$

$\rightsquigarrow$ *dispersionless KdV hierarchy* $\rightsquigarrow$ *Witten Conjecture.*

Example 2. [*W*(*M*)=*extended affine Weyl group* $\widetilde{W}(A_1)$]

Quantum cohomology of $\mathbb{CP}^1$:

$F=\frac{1}{2}(t^1)^2t^2+e^{t^2}$, $E = t^1\partial_1+2\partial_2$, $e = \partial_1$.

$\rightsquigarrow$ *dispersionless extended Toda hierarchy* $\rightsquigarrow$ *Toda Conjecture.*

It might be conjectured that for a Frobenius manifold with good analytic properties the monodromy group acts discretely in some domain of the space. The Frobenius manifold itself can be identified with the orbit space of the group in the sense to be specified for each class of monodromy groups. A natural question is how to construct the Frobenius manifold structure on the orbit space of some groups. Equivalently, *"Which kind of groups can be realized as monodromy groups of Frobenius manifolds?"*

It was shown by B.Dubrovin in 1993 that any finite Coxeter group can serve as a monodromy group of a polynomial Frobenius manifold, that is to say, the potential F(t) is a polynomial w.r.t $t = (t^1,\cdots,t^l)$. Furthermore, B.Dubrovin put forward the following conjecture,

"Any massive polynomial Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group",

which was proved by C.Hertling in 2002. More generally, we have shown that there are l different Frobenius structures on the orbit spaces of the Coxeter groups B_l and D_l. Especially, these are meromorphic along the divisors $t^k = 0$ for $k = 1,\cdots, l-2$.

Motivated by the Example 2, in 1996 B.Dubrovin and Y.Zhang introduced a certain extensions of affine Weyl groups, called extended affine Weyl groups. It was shown that for some particular choices of the simple root α_k, a Chevalley type theorem holds true for the ring $\mathcal{A}$, and thus the orbit space defined as $\mathcal{M}$= Spec$\mathcal{A}$ of the extended affine Weyl group $\widetilde{W}^k$ is an affine algebraic variety of dimension l + 1. It was further proved that on such an orbit space there exists a

Coxeter groups

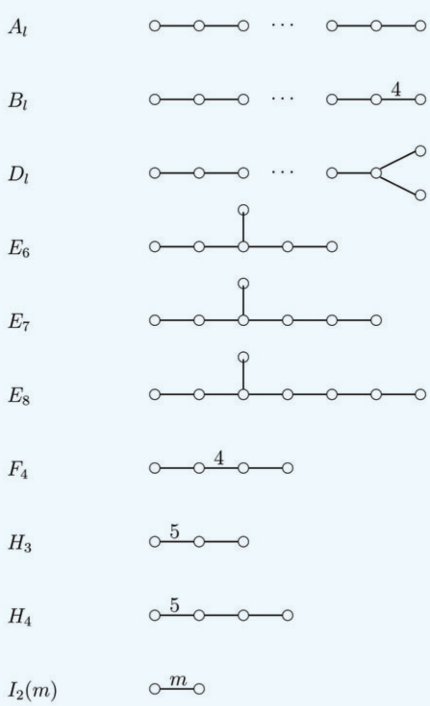

Table 1

Frobenius manifold structure whose potential is a weighted homogeneous polynomial of t^1, $\cdots$, t^l, t^{l+1}, where t^1, $\cdots$, t^{l+1} are the flat coordinates of the Frobenius manifold. Similarly a natural conjecture, due to B.Dubrovin and Y.Zhang, is

"Any massive weighted homogeneous polynomial Frobenius manifold, that is to say the potential F(t) is a polynomial of t^1, $\cdots$, t^l, $e^{t^{l+1}}$ *with positive invariant degrees except* $deg(t^{l+1}) = 0$, *is isomorphic to the orbit space of a particular extended affine Weyl groups?'*

B.Dubrovin and Y.Zhang have checked this for $l \leq 3$. For general l this is still open.

Notice that for the root system of type A_l, there is in fact no restrictions on the choice of α_k. However, for the root systems of type B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2 there is only one choice for each. In 1997 Slodowy pointed out that the Chevalley type theorem is a consequence of the results of Looijenga and Wirthmüller, and in fact it holds true for any choice of the base element α_k, or equivalently, for any fixed vertex of the Dynkin diagram. A natural question is

"Whether the geometric structures that were revealed by Dubrovin-Zhang's construction also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of α_k*?"*

Our recent work is to give an affirmative answer to this question for the root systems of type B_l, C_l and also for D_l. We show, by fixing another integer $0 \leq m \leq l-k$, that on the corresponding orbit spaces there also exist Frobenius manifold structures with potentials $F(t)$ that are weighted homogeneous polynomials w.r.t $t^1, \cdots, t^{l-1}, t^l, \frac{1}{t^{l-m}}, \frac{1}{t^l}, e^{t^{l+1}}$.

It remains a challenging problem to understand whether the constructions can be generalized to the root systems of the types E_6, E_7, E_8, F_4, G_2 with respect to the choice of an arbitrary vertex on the Dynkin diagram. Another question is to generalize the above constructions to more general extended affine Weyl groups. This work is in progress.

The another direction is to study the Frobenius structures related to complex crystallographic Coxeter groups $\widetilde{W}(\tau)$ which was introduced by J.Bernstein and O.Schwarzman, implicitly also by E.Looijenga. J.Bernstein and O.Schwarzman also found an analogue of the Chevalley theorem for $\widetilde{W}$. In order to construct the Frobenius structure it needs a $\widetilde{W}(\tau)$- invariant metric with some particular property. Unfortunately such metric does not exist. To resolve this problem, B.Dubrovin considered a certain extension of $\widetilde{W}(\tau)$, i.e. the Jacobi group $\mathbf{J}(\mathcal{G})$ which was proposed by M.Eichler and D.Zagier. Here $\mathcal{G}$ is a simple Lie algebra with Weyl group W. The corresponding orbit space is called the Jacobi form $J(\mathcal{G})$. B.Dubrovin in 1993 studied the Frobenius structure related to the Jacobi forms $J(A_1)$. Afterwards, M.Bertola in 2001 generalized this construction and obtained a Frobenius manifold structure on $J(\mathcal{G})$ for $\mathcal{G}=A_l, B_l, G_2$. Especially, in the case of type G_2, M.Bertola can find two different Frobenius structures.

Extended affine Weyl groups

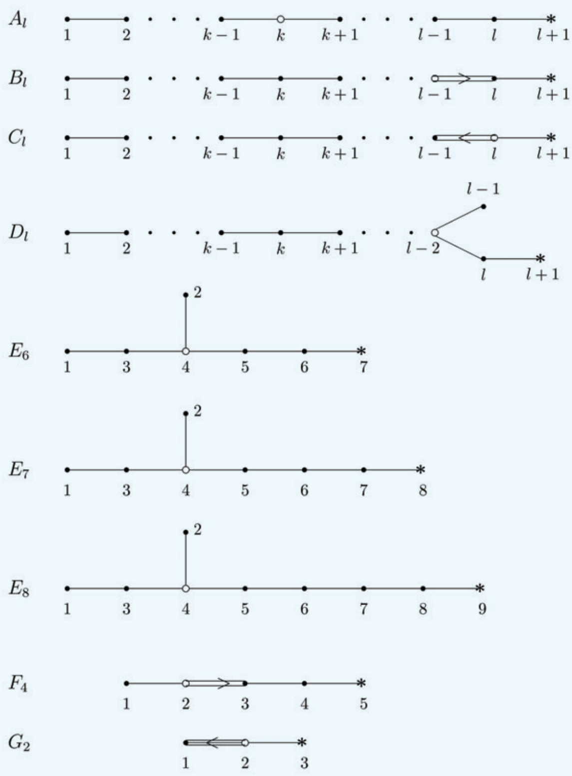

Table 2

The first structure comes from the embedding of $J(G_2)$ in $J(A_2)$, while the second was constructed by K.Saito. In 1993 and 1998 I.Satake studied the Frobenius structures on $J(\mathcal{G})$ for $\mathcal{G}=E_6, D_4$. The recent work of I.Satake gave a complete study for general cases. He showed that there was a Frobenius structure on the complex orbit space of the reflection group for an elliptic root system of codimension 1. KIAS